XVIII. *Propositions containing some Properties of* TANGENTS *to* Circles; *and of* TRAPEZIUMS *inscribed in Circles, and non-inscribed. Together with Propositions on the* Elliptic *Representations of* CIRCLES, *upon a plane surface, by* PERSPECTIVE.

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THE propositions of the first series, and which are termed Introductory, had their origin from my attention being directed to the perspective representation of a Circle. But they are kept separate from the perspective propositions. Comprising some things believed to be new, and forming a short system in themselves, they may, on these accounts, be acceptable to other Readers besides those who study perspective.

A desideratum, in the science of perspective, had been suggested to me by the Reverend Mr. KERRICH, Principal Librarian to the University of Cambridge; namely, of some *law* or *laws* which the axes, of ellipses representing circles, might be found to observe in their directions. An attempt, grounded on this suggestion, is contained in the four Propositions concluding the second or Perspective series. If these be correct, it is hoped that a foundation is laid, for further improvement by persons younger than myself and differently circumstanced. And some mechanical instrument * may

* The invention of an Instrument for the ready delineation of Ellipses, by Mr.

possibly be brought to co-operate with the theory here given or begun, so as that, in conjunction, they may afford considerable advantage to the practice of perspective; especially where a number of circles together are to be represented.

The same four propositions are built upon the sixth on perspective: of which the construction (as given in fig. 5.) may be seen in HAMILTON'S* Stereography. That it was not taken from his work, but separately discovered, has no claim to the attention of the reader, except in this respect; that, in the solutions of any problem, the more persons there are who give independently the same solution, the more likely it is (cæteris paribus) to be better than any solution not yet discovered. The *demonstration* also has been varied; both of this sixth proposition, and of some other things which, though not new in substance, it has been judged expedient to insert.

For the seventh of the propositions on perspective, I am also indebted to Mr. KERRICH: though, for the proof here given of it, I am responsible.

The foregoing account appears to be all the previous information necessary to trouble a reader with, who may be going to examine the propositions in his closet. But, as mathematical demonstrations, accompanied with diagrams, cannot be accommodated to a public reading, some further particulars are therefore here subjoined: which, if the Paper should have the honour of being received by the Royal Society, may give a better insight into its nature and intent.

The whole had its origin in the study of that part, of the

John Farey junior, obtained the gold medal, in May 1813, from the Society for the Encouragement of Arts. For its merits I refer to that Society.

* B. III. Sect. II. Prob. 1.

science of perspective, which relates to the representation of a *circle*. And the principal result (so far as immediately respects that science) is contained in the four propositions which conclude the paper, together with the lemma which precedes them. This is called a lemma rather than a proposition, because it does not necessarily involve the consideration of a circle.

That the perspective representation of an entire circle is an ellipse or circle, is simply a part of the doctrine of conic sections. To that doctrine therefore it is referred, in the first of the perspective propositions. An *entire* circle, supposed to be viewed by the eye through the plane of the picture, excludes (by the hypothesis) the two cases in which the circle touches and cuts the plane, passing through the eye, parallel to the picture. In which two cases, the representation is a parabola and hyperbola respectively. Not having any thing new to offer upon these representations, or not any thing appearing likely to improve the practice of perspective, I have omitted the two cases entirely; and confined myself to the *elliptic* representations, the circular included.

The principal inquiry has been, whether, in any instances of a number of circles, lying in one plane and having their centres in one right line, any *law* or *laws* could be discovered to be observed by the major or minor axes of the representing ellipses, in their directions or positions. If it could have been proved that the axes converged to a known or easily discoverable point, this would have been a great acquisition to the science of perspective, and have given a direct facility to the practice. But, if indeed they do *not* so converge, it is natural to wish that this should be demonstrated. Such a demonstration puts an end to further inquiry about the point to which they might be imagined to converge, and wholly about such convergency in them; and leaves us free to inquire after any other law or regularity which the axes *may* observe.

Now the object of the lemma (with its first corollary), prefixed to the eighth of the propositions on perspective, is to prove that lines, in a picture, cannot converge to a point; unless the original lines which they represent happen so to converge or else to be parallel. The inference is, that the axes of the ellipses cannot so converge, unless the chords which they represent should so converge or be parallel: a case which may rarely happen, and, if happening, may not be readily discernible.

The convergency to a point, by the axes, being therefore dismissed, the four remaining propositions, after the lemma, exhibit nevertheless a regularity in the directions of the axes. Though these do not converge to a point, they are (if the reasoning be just) parallel respectively to other lines which *do* converge to a point. To find each of these other lines separately and accurately, is not the object proposed. But certain *limits* are pointed out, within which the artist is to keep; and a regularity marked, by which the variation in the directions of successive axes is to be guided: which, it is hoped, will produce a sufficient accuracy in a great number of instances. And, where any one or more circles require singly a strict accuracy, to this the construction of the sixth proposition is adapted.

How much, or whether any part, of the first seven propositions on perspective, might have been omitted, without injury to that series of propositions, I do not take upon me to decide. Certainly much of the matter, contained in the seven, is not new. But, so far as it is necessary to the principal design, its admission will be justified. And also new *proof*, of what is already known, is sometimes acceptable to the mathematician, and may be admitted as an advancement of science. On which ground the proof here given of the sixth proposition, and some other proofs, await the decision of the reader.

An indirect result of the research, is merely geometrical and may be taken separately from perspective. This consists of the introductory propositions. On the one hand, it appeared necessary, or desirable, that these should accompany the perspective propositions; because of various references, in the latter, to the former. On the other hand, it seemed also desirable that the introductory propositions should be so far kept separate, as to enable any reader to peruse them, who would have declined all examination of their contents if interwoven with perspective.

The properties of the circle, not improbably, lie open to abundant successful investigation. It is hoped that the introductory propositions, of this paper, may have contributed a little to the stock of knowledge respecting that remarkable geometrical figure.

To avoid crowding the plates with figures, I have, in a few instances, left it to the reader to construct his own figures for illustration, if they appear to be wanted. For instance, in cor. 6 to introd. prop. X, and cor. 1 to introd. prop. XII.

INTRODUCTORY PROPOSITIONS.

Some Properties of TANGENTS *to Circles; and of* TRAPEZIUMS, *inscribed in Circles, and non-inscribed.*

Definitions, for the Introductory Propositions.

Def. 1. The *tangent-chord*, of a given point without a circle, is the chord from whose extremities tangents meet in the given point.

Def. 2. A *vertex* to a base is any point, within a circle, through which are drawn chords: and the right line joining the points to which those are the tangent-chords, is the *base* to that vertex.

Def. 3. The *mean point* is a point in the diameter perpendicular to such base: and its distance from the base is a mean proportional between the distances of the extremities of that diameter, from the base.

Thus (Fig. 1): If the chords PQ, GH, pass through V within a circle, and if PD, QD, GL, HL, be tangents, and LD be drawn; PQ is (def. 1) the *tangent-chord* of D, GH that of L; and (def. 2) V is the *vertex* to the *base* LD. If a diameter FB produced cut LD perpendicularly in D, and if, in DF, be taken DM equal to DP, therefore* a mean proportional between DF and DB, M is (def. 3) the *mean point*.†

Prop. I. Fig. 1, 2. In a given circle, one vertex has only

* III. Eucl. prop. 36, and VI. 14. Simson's edition, 1762, is referred to throughout.

† Other Definitions after Prop. VII.

one base; a right line indefinite in length: and one base has one vertex.

Thus: if GH and any other chords pass through the vertex V; the tangents from their extremities meet respectively in points of one indefinite line LD: and, conversely, if tangents be drawn from L and other points of LD; all their tangent-chords pass through one point V.

Demonstration. Fig. 2. Let E be the centre, and PQ the chord bisected in V. Draw the tangents GL, HL, PD, QD; and draw DE, DL, LE, EP, EQ, EH. On the diameters DE, LE, describe circles. That on DE passes through P and Q, because of the right angles DPE, DQE; and that on LE through G and H. And DE passes through V, because of the similar and equal right-angled triangles DEP, DEQ. By the circle on DE, the rectangle DVE is equal to PVQ; which, by the given circle GPQ, is equal to GVH; whence GEHD is* a circle, and is that on LE, and LDE is† a right angle. In like manner, if any other chord ST pass through V, and if tangents from S and T would meet in R, RDE would be proved a right angle. Therefore LDR is one right line. And conversely (fig. 1), if L be any point in a base to V; its tangent-chord GH passes through V. If not, let another chord G*h* so pass. Then (as proved) the tangent from *h* meets GL in that base, therefore in L; that is, in the same point as the tangent from H. Which is impossible. Therefore GH passes through V. *Q. E. D.*

Cor. 1. Fig. 1. The indefinite base to a given vertex V, is a line parallel to the chord bisected in V, and passing through any point, L or D, in whose tangent-chord V is.

* By a necessary converse of III Eucl. prop. 35. † III Eucl. prop. 31.

And the vertex to a given base LR, if E be the centre and ED a perpendicular on LR, is the intersection of ED with the tangent-chord of any point D or L of the base.

Cor. 2. Fig. 1. The mean point M, to a given base, is in the same diameter with V: and DM, as equal to DP, is a mean proportional between DE and DV. For, if EP be drawn, the triangles EDP, PDV, are similar.

Cor. 3. Fig. 2. If a chord GH bisect PQ the tangent-chord of any point D; DE bisects the angle GDH. For, draw DG, DH, EG. The right lines EG, EH, are equal; and the arcs EG, EH; whence the angle GDE is* equal to EDH. And, conversely, if DE bisect GDH, GH bisects PQ the tangent-chord of D. For, if any other chord through G should bisect PQ, the same angle GDE would be equal to one greater or less than EDH: which is impossible.

Cor. 4. Fig. 2. V is the vertex to the base LR in every circle whose centre, as E, is in DV produced beyond V, and whose radius, as EP or EQ, is found by the intersection P or Q of a circle on the diameter ED with the parallel PQ. For DPE and DQE are† right angles; whence DP and DQ are‡ tangents.

Prop. II. Fig. 3. The distance of an assumed point, in a given base, from the mean point within the circle, is equal to a tangent from the assumed point.§

Thus: if L be assumed in the base LD, and if M be the mean point within the circle GBQ, and LG a tangent; LM is equal to LG.

* III Eucl. prop. 27. † III. Eucl. prop. 31. ‡ III. Eucl. prop. 16. Cor.

§ This is a lemma in Hamilton's Stereography, with a different proof. B. III. Sect. I. L. 18.

Dem. From the centre E draw ED perpendicular to LD; and draw the tangent DQ; also EQ, EG, LE. The squares of LM and radius are equal to those of LD and DM (or DQ) and radius; therefore to those of LD and DE, so to that of LE or those of LG and radius; whence the square of LM is equal to that of LG, and LM to LG. *Q. E. D.*

Prop. III. Fig. 4. If the tangent-chord of any point will, produced, pass through any other given point; that of the second will pass through the first; and the squares of the two tangents, one from each point, are together equal to that of the intercepted right line; and, such line being taken as a base, and the mean point to it being found, lines to this from the two points contain a right angle.

Thus: if ST, the tangent-chord of R, will pass through L; GH, that of L, will pass through R; and the squares of the tangents LG and RS are together equal to that of LR; and, LR being a base, LMR at the mean point is a right angle.

Dem. Draw LH; and let it be between LR and LG. Draw from the centre E, to LR, a perpendicular ED; also GD, GR. Describe the circles LGEH, RSET *. These will pass through D, because of the right angles LDE, RDE; whence the rectangle RLD is equal to SLT, so to the square of LG, and LD is to LG as LG to LR. Therefore the angle LGR is † equal to LDG, to LHG, to LGH; and GH produced will pass through R.

And, because of the circle GHD, the rectangle LRD is equal to GRH, so to the square of RS; whence this square and that of LG (proved equal to RLD) are ‡ together equal to that of LR.

* As LGEH in Prop. I. † VI. Eucl. prop. 6. ‡ II. Eucl. prop. 2.

And LM is * equal to LG, and RM to RS; whence the squares of LM and RM are together equal to that of LR. Therefore † LMR is a right angle. *Q. E. D.*

Cor. 1. If V be the vertex to LR, and right lines from L and R to E, also to V, be drawn or conceived; the angle LER is acute, LVR obtuse. ‡

Cor. 2. If the tangent-chord of a point L will pass through a point R, and if D in LR make RLD equal to the square of LG; ED is perpendicular to LR. For, draw EH. Then RL is to LG as LG to LD; whence the angle LDG is § equal to LGR or LGH, so to LHG, and LGEHD is one circle. But LHE is a right angle; therefore also LDE.

Prop. IV. Fig. 4. If two points be such, that, the intercepted right line being a base and the mean point to it being found, lines from this to the given points contain a right angle, or that the squares of the tangents from the two points are together equal to that of the intercepted line; the tangent-chord of each of the two points will pass through the other.

Thus: if, LR being a base, LMR at the mean point be a right angle, or the squares of LG and RS be together equal to that of LR; the tangent-chord of L will pass through R, and that of R through L.

Dem. On the first supposition, the tangent-chord of R will pass through L: else it would cut the indefinite base in some other point, from which a right line to M would be || perpendicular to MR; which is impossible. Therefore also ¶ the tangent-chord of L will pass through R.

On the second supposition, LMR will be ** a right angle;

* Prop. II. † I Eucl. prop. 48. ‡ I Eucl. prop. 21.
§ VI Eucl. prop. 6. || Prop. III. ¶ Ib. ** *Dem.* of Prop. III.

whence (by the present proposition) the tangent-chord of each of the two points will pass through the other. *Q. E. D.*

Cor. If either LMR be a right angle, or the squares of LG and RS be together equal to that of LR; the other of these two follows from it.*

Prop. V. Fig. 5. If a diameter be cut by a perpendicular from any point of the circumference, and, produced, be cut by a tangent from the same point, and if an angle be made at any point of the circumference, by right lines to the two intersections; this angle and its supplement are bisected by right lines to the extremities of the diameter.

Thus: if LR, a diameter of GNH, be cut in I by NI, and, produced, in K by NK, and if GK cut the circumference in H between G and K; the angles INK, IHK, IGK, are bisected by NR, HR, GR, and their supplements INX, IHY, IGY, by NL, HL, GL; KN being produced to X, and KG to Y.

Dem. Produce NI to O in the circumference, and draw OR, OL. Then RNK is † equal to NOR, so to ONR or INR; also LNX to NOL, so to ONL or INL. Produce GI, HI, to T, S, in the circumference. Draw HT, SK, SR, SG, SL. Then, since HS and GT bisect NO, HKR or GKR is ‡ equal to RKS or RKT, HK § to TK, GK to SK; and HT and GS are perpendicular to LR. IHR or SHR is equal to SGR, so to GSR, which, having the supplement || RHG, is equal to RHK. And, from the equal arcs RH, RT, TGR or IGR is equal to RGH or RGK. And the right lines LG, LS, are equal, and the arcs LG, LS; whence LHS or LHI is equal to LHG or LHY.

* Prop. III. and IV.

† III EUCL. prop. 32.

‡ Cor. 3 to prop. I.

§ III EUCL. prop. 8.

|| III EUCL. prop. 22.

And LGI (LGS and IGS) is equal to LSG and ISG together, that is, to LSI, which, having the supplement LGH, is equal to LGY. *Q. E. D.*

Cor. 1. The angle HIT is bisected by LR, which bisects HT perpendicularly: and GIH is bisected by NO, because HIR is equal to TIR, so to GIL.

Cor. 2. In every triangle in which IK is one side, and the opposite angle is at the circumference of the circle, the two other sides, adjacent to IR and RK respectively, are * in the ratio of IR to RK.

Cor. 3. Fig. 6. If a right line IK be divided unequally in R, in a ratio given; a circle LNR may be found such, that, IK being a side of any triangle having the opposite angle at the circumference, one of the other sides shall bear to the remaining side the given ratio. Describe the semicircle ITK, draw RT perpendicular to IK, and the tangent TL cutting KI produced in L: a circle LNR, on the diameter LR, is the circle required. For, let the two circumferences cut each other in Q; bisect IK in F, LR in C; draw IN perpendicular to LR; also FQ, CQ, FT, CN, KN. Then, since LTF and TRF are right angles, the rectangle LFR is † equal to the square of FT, so of FQ; whence FQ touches ‡ LNR, FQC is a right angle, and CQ touches § ITK; whence the rectangle KCI is || equal to the square of CQ, so of CN, and the angle KNC, as equal ¶ to NIC, is a right angle, therefore KN a tangent.

Prop. VI. Fig. 5. If the tangent-chord of a given point be

* VI Eucl. prop. 3. † VI Eucl. prop. 8. Cor. and prop. 17.
‡ III Eucl. prop. 37. § III Eucl. prop. 16. Cor. || III Eucl. prop. 36.
¶ VI Eucl. prop. 14 and 6.

cut by a chord produced to that point; the produced chord is divided in harmonical proportion.*

Thus: if NO the tangent-chord of K be cut in V by a chord GH, or in I by the diameter LR, each, when produced, passing through K; then GK is to HK as GV to VH, and LK to RK as LI to IR. That is; the distances of G and H, of L and R, from K, are proportional to their distances from V and I respectively.

Dem. Let the figure be as in Prop. V.; and let GA, HB, be perpendiculars on LR. Then the angles HIR, GIL, are † equal. Therefore the triangles HIB, GIA, are similar; and GK is to HK as GA to HB, as AI to IB, as GV to VH. Further. LK is to RK as the rectangle LKR or the square of NK to the square of RK, as that of ‡ NI to that of IR, as § LI to IR. *Q. E. D.* ||

Cor. 1. If a chord GT, through any vertex I, will, produced, cut KC, the indefinite base to I, in any point C; then GC is to TC as GI to IT, and GC to GI as TC to TI. For the tangent-chord of C passes ¶ through I.

Cor. 2. If GT be a chord through any vertex I, and Z any point in the base, and if the chord bisected in I be cut in D by ZG, in W by ZT produced; ID is equal to IW. For DW

* This proportion has been expressed in different ways. A line LK consists of three parts, LI, IR, RK; and the whole line is to either extreme part, as the other is to the middle part. Otherwise: LK, LR, LI, are three lines; and the first is to the third, as the difference of the first and second is to that of the second and third.

† In Cor. 1 to prop. V.

‡ Cor. 2 to prop. V.

§ VI Eucl. prop. 8. Cor. and prop. 20. Cor. 2.

|| Among the proofs which might be given, of this proposition, one (applicable both to GH and LR) is by VI Eucl. prop. 3 and prop. A following it, together with the Prop. V here given and its first corollary.

¶ Prop. I.

is parallel to KC: and CZ is to ID as GC to GI, as * TC to TI, as CZ to IW. And, conversely, if ID be equal to IW, then GD and WT, produced, will meet in a point Z of the base.

Cor. 3. Fig. 6. If L, I, R, K, in that order, be points in one right line, and LK be to RK as LI to IR, and if LR be a diameter of a circle LNR, and IN perpendicular to LR; KN is a tangent. For, if not, the tangent-chord of K will cut LR in some point X other than I. Then LX is to XR as LK to RK, as LI to IR: which is impossible. So, if IK be the diameter of ITK, and RT a perpendicular; LT is a tangent.

Cor. 4. Fig. 6. If any three of L, I, R, K, be given; the fourth, if an extreme point, as K, may be found by LNR, IN, NK; if an intermediate point, as R, then by ITK, LT, TR.

Cor. 5. Fig. 7. A circle PHF may be described, having a given vertex V to a given base LR, and passing through a given point P whose perpendicular distance from LR is greater than from VF parallel to LR. Let GD be a perpendicular, through V, on LR. Draw PD, PV; also PG, making the angle VPG equal to PDG. G is the centre required. For, draw GF. Then, by the similar triangles DGP and PGV, the rectangle DGV is equal to the square of PG or GF; whence the triangles DGF and FGV are similar, DFG is a right angle, as equal to FVG, and DF a tangent; whence V is † vertex in PHF to LR. Let DV cut the circle first in H, then in K. DK is to KV as DH to HV. Therefore DH is greater than HV; and, if P were not further from LR than from VF, the thing could not be done. If P be given in the

* Cor. 1.

† Cor. 1 to prop. I.

perpendicular, as at H or K, the circle is found from the proportion, DK to KV as DH to HV.

Cor. 6. Fig. 7. If two circles, SQT and HFK, have the same vertex V to a given base LR; they are not concentric, and one includes the other and does not touch it. Thus: if E be the centre of SQT, and G of HFK, and EV be greater than GV; SQT includes HFK, and does not touch it. For, let VD be the perpendicular on LR. Draw VQ parallel to LR, cutting SQT in Q and HFK in F. Draw DQ, DF: which are tangents. Then, if any point X were the common centre of the circles, and XF and XQ were drawn, both XFD and XQD would be right angles: which is impossible. E and G are * in DV produced: which line will cut SQT first in a point S, then in a point T; and in like manner cut HFK, in H and K. Then DK is to KV as DH to HV, and DT to TV as DS to SV. But DK is less than DT; whence the ratio of DK to KV is greater than of DT to TV. Therefore that of DH to HV is greater than of DS to SV; whence SV is greater than HV. *A fortiori* the whole of SQT falls without HFK.

Prop. VII. Fig. 7. *Problem.* Two circles, not concentric, being given, of which one includes the other and does not touch it; to find a base, having a vertex common to the two circles.

Solution. Let E be the centre of the outer circle SQT, and G that of the inner HFK. Draw EG indefinite; cutting the circles in H and S beyond G, in K and T beyond E. Find a right line A equal † to $\frac{sq.\,ES - sq.\,GH}{EG}$ — EG; and divide it ‡

* Cor. 1 to prop. I. † See I EUCL. prop. 47; and VI EUCL. prop. 17 and 11.

‡ Cut it perpendicularly by a chord equal to 2GH, in a circle on the diameter A.

into two parts whose rectangle shall be equal to the square of GH. Make GV equal to the less part, and GD to the greater; taking V and D in EG produced beyond G. Through D, perpendicular to GD, draw LR the base required: to which V is the common vertex.

Dem. First, A is greater than 2GH. For sq. ES — sq. GH is greater than sq. EH — sq. GH, which is * sq. EG + 2EGH; whence A is greater than $\frac{sq.\ EG + 2EGH}{EG}$ — EG, which is 2GH. Therefore A can be divided as required.

Through V draw VQ parallel to LR; cutting HFK in F, and SQT in Q. Draw GF, FD, EQ, QD. Then, because the rectangle DGV is equal to the square of GH or GF, the triangles DGF and FGV are similar, the right angle GVF is equal to GFD, DF is a tangent, and V the vertex † in HFK to the base LR.

Further. To A (DG + GV) add EG. Then $\frac{sq.\ ES - sq.\ GH}{EG}$ is equal to DE + GV; whence sq. ES = DEG + EGV + sq. GH (DGV), = DGE ‡ + sq. EG + EGV + DGV, = $\overline{DG + EG} \times \overline{EG + GV}$, = DEV. Since, then, the square of ES or EQ is equal to DEV, DQ (as DF before) is a tangent, and V the vertex in SQT to LR. *Q. E. D.*

EV and ED might have been found, instead of GV and GD; *mutatis mutandis:* by making A equal to $\frac{sq.\ ES - sq.\ GH}{EG}$ + EG, and the rectangle under its parts equal to the square of ES.

* II Eucl. prop. 4. † Cor. 1 to prop. I. ‡ II Eucl. prop. 3.

Additional Definitions for the Introductory Propositions.

Def. 4. Fig. 8. The *points of concourse*, of a *trapezium*, are the two points in which (if at all) the opposite sides, produced, meet respectively. Thus: if ABCF be a trapezium, and if FA and CB meet in L, AB and FC in R; the points of concourse are L and R.

Def. 5. The *connecting line* is a right line joining the two points of concourse; as LR. And, if the figure be one inscribed in a circle, LR is cut in the *dividing point* D by a perpendicular from the center.

Def. 6. The *near* angle and the *remote* angle, of a trapezium having two points of concourse, are those, respectively, whose angular points have the least, and the greatest, perpendicular distances from the connecting line. The two others are the *mean* angles.

Prop. VIII. Fig. 8. The near angle, of an inscribed trapezium having two points of concourse, is greater than any of the others; and the remote angle is less than any.

Thus: if the figure be as in def. 4, and ABC be the near angle, AFC the remote; ABC is greater than each of BAF, BCF, AFC; which last is less than each of BAF, BCF.

Dem. BAF, having * the supplement BCF, is equal to RCB; and BCF to LAB. Therefore ABC, an exterior angle of the triangles RCB and LAB, is greater than either of the mean angles at A and C: and their equals, RCB and LAB, being exterior angles of the triangles LFC and RFA, are severally greater than AFC. *Q. E. D.*

* III Eucl. prop. 22.

Cor. 1. ABC is always obtuse, and AFC acute; because their sum is two right angles: whence AC is not a diameter.

Cor. 2. BF, subtending the mean angles, is greater than the other diagonal. For, if they be right angles, BF is a diameter. If they be not, one is acute; as BAF. Then, since this is greater than AFC, the double of it, that is, an angle on BF at the* centre, is greater than one on AC at the centre; therefore† BF than AC.

Cor. 3. The greater diagonal, produced, cuts LR between L and R; the less beyond L or R, if at all.

Cor. 4. The sides containing the remote angle, are greater than their opposites; FC than AB, AF than BC. For the triangle LAC is similar to LBF, and LCF to LAB. Therefore LF is to LC as BF to AC; whence LF is‡ greater than LC, and than LB. But LF is to LB as FC to AB. By the like proof, AF is greater than BC.

Prop. IX. Fig. 9. The tangent-chord of each point of concourse, of an inscribed trapezium, will, produced, pass through the other.

Thus: if L and R be the points of concourse; the tangent-chord of L will pass through R, and that of R through L.

Dem. Let ABCF be the trapezium, ABC the near angle, AFC the remote, and L the concourse of FA and CB. Describe the circle ABL, meeting LR in D; and draw AD, also the tangents LG, RS. Then the angle LDA is equal to LBA, which, by the common supplement ABC, is equal to LFC or LFR; whence the triangle LDA is similar to LFR, and the rectangle RLD equal to FLA, so to the square of LG. But LRD is equal to ARB, so to the square of RS. Therefore

* III Eucl. prop. 20. † I Eucl. prop. 24. ‡ Cor. 2.

the squares of LG and RS are together equal to RLD and LRD, so to the* square of LR; whence the tangent-chord of L will† pass through R, and that of R through L. *Q. E. D.*

Cor. 1. If E be the centre, V the vertex to LR taken as a base, and M the mean point, and if right lines be drawn or conceived, from L and R, to E, to V, to M; LMR is a right angle, LVR obtuse, LER acute.‡

Cor. 2. D is the dividing point.§

Cor. 3. Each side of ABCF is a chord in a circle passing through the opposite point of concourse and the dividing point. For, describe the circle FCL; and let D be now the point in which this circumference cuts LR; and draw DF. Then the angle LDF is equal to LCF, so to LAR; whence the triangle LDF is similar to LAR, and the rectangle RLD equal to FLA, and so to the square of LG, as before. Therefore D, as|| before, is the dividing point. And a circle CBR would be proved as ABL, and FAR as FCL, to pass through that point.

Cor. 4. The diameter LI of ABL is perpendicular to FC. For, let them meet in K; and draw ID. Then the angle LID is equal to LAD, so to LRF or LRK; whence LKR, as equal to LDI, is a right angle. In like manner the diameter of FCL would be proved perpendicular to AB, that of CBR to AF, that of FAR to BC.

Prop. X. Fig. 9. If an inscribed trapezium have two points of concourse; the connecting line is the base to which the mutual intersection of the diagonals is vertex in the inscribing circle.

Thus: if ABCF have L and R its points of concourse; LR

* II EUCL. prop. 2. † Prop. IV. ‡ Prop. III, and its Cor. 1.
§ Def. 5, and Cor. 2 to prop. III. || This Prop. and Cor. 2.

is the base to which V, the intersection of AC with BF, is vertex in the circle ABCF.

Dem. Let ABC be the near angle, AFC the remote, E the centre, D the dividing point, and A in LF. Draw ED, DA, DC, DB, DF. Draw or conceive the circles* ABDL &c. Then, by ABDL, the triangle LDA is † similar to LFR; and, by CBDR, RDC to RFL: whence the angles LDA, RDC, are equal, and ADC is bisected by DE. Further. If BF be a diameter, it coincides with ED; which ‡ bisects the tangent-chord of D. Let it be not a diameter. By the circle ABDL, the angle LDB has the supplement LAB, and is equal to BAF; and, by FADR, RDF is equal to RAF or BAF: whence LDB, RDF, are equal; also LDF, RDB; and BDF is bisected by DE. Therefore the tangent-chord of D is § bisected by AC and BF, in V. But ED bisects it: whence V is its intersection with ED, therefore || the vertex to LR. *Q. E. D.*

Cor. 1. Fig. 5. If two sides, as GS and HT, be parallel; a line KC parallel to them is the base. For, let GH and ST meet in K, GT and SH in I. Then one diameter LR bisects GS and HT perpendicularly, and, produced, bisects the angle GKT or HKS; whence GT and SH bisect ¶ the tangent-chord of K in I: which is perpendicular to LK or LR, and so is parallel to GS and HT; whence KC is ** base to the vertex I. Also GH and ST are equal.

Cor. 2. Fig. 9. DAEC, DBEF, are circles; except BF be a diameter of the inscribing circle. For the rectangle DVE

* Cor. 3 to prop. IX.

† *Dem.* of prop. IX.

‡ *Dem.* of prop. I.

§ Cor. 3 to prop. I.

|| Cor. 1 to prop. I.

¶ Cor. 3 to prop. I.

** Cor. 1 to prop. I.

is equal to the square of* half of the tangent-chord of D; therefore to AVC, and to BVF.

Cor. 3. Fig. 10. If ABCF be an inscribed trapezium, having V the mutual intersection of the diagonals, L and R the points of concourse, ABC the near angle, AFC the remote, and A in LF, and if FB and AC, produced, cut LR in I and K; LK is to RK as LI to IR. For, let LB and RB, produced, cut SP, drawn through V parallel to LR, in P and S. Then LK is to PV as CK to CV, as† AK to AV, as RK to SV; whence LK is to RK as PV to SV, as LI to IR.

Cor. 4. Fig. 10. If to M, the mean point to the base LR, be drawn LM, IM, RM, and KM produced to any point X; the angle IMK is bisected by MR, and IMX by ML. For, on the diameter LR, and on the same side with the trapezium, describe the semicircle LNR, and draw IN perpendicular to LR. Then KN, being drawn, is‡ a tangent. And, because LMR is§ a right angle, M is in the circumference LNR; whence|| the angles are bisected.

Cor. 5. Fig. 10. Or, if a base LK be cut in any two points I and K by chords, as FB and AC, through its vertex V, and M be the mean point; the angle IMK is bisected by drawing either AB or FC, producing it to R in the base, and drawing MR. For AB and FC meet the base in one point: and the inscribed trapezium ABCF may be completed. Also IMX might be bisected in a similar manner.

Cor. 6. Fig. 9. If any polygon, having an even number of sides, be inscribed in a circle, and its diagonals joining opposite¶

* Of PV in Fig. 2. † Cor. 1 to prop. VI. ‡ Cor. 3 to prop. VI.
§ Cor. 1 to prop. IX. || Prop. V.

¶ By *opposite* angles and sides are meant such, that the two intercepted arcs contain equal numbers of angular points.

angles meet in one point V, not the centre; the opposite sides will respectively meet, if at all, in points of the base to the vertex V. For any two opposite sides, as AB and FC, may be made the opposite sides of an inscribed trapezium having the same intersection of its diagonals. If two opposite sides be parallel, the base is * parallel to them.

Cor. 7. Fig. 11. If ABCF and LR be as in the proposition, and if IK, KN, NO, OI, be tangents respectively at A, B, C, F; the diagonals IN, KO, of the circumscribed figure IKNO, cut each other in the same point V as do those of ABCF, and are in the same indefinite lines as the tangent-chords of L and R. For, since AB the tangent-chord of K will pass through R, that of R will † pass through K; likewise through O, whose tangent-chord is FC. And it will pass through ‡ V, and § L. In like manner the tangent-chord of L will pass through N, I, V, and R.

Cor. 8. Fig. 11. Therefore the opposite sides of a circumscribed trapezium, if they meet, meet in points of the same indefinite base as that of such inscribed one; namely, whose angular points are the points of contact of the circumscribed. For the sides are tangents from chords through V. If two opposite sides be parallel, the base is parallel to them. For, if IO and KN, which are tangents at F and B, were parallel, FB would be a diameter, and perpendicular to them: and, because it passes through V, it would be the diameter perpendicular to LR.

Cor. 9. Fig. 11. Also, if a circumscribed polygon have an even number of sides, and its diagonals || pass through one

* Cor. 1. † Prop. III. ‡ Prop. I.
§ Prop. IX. || As in Cor. 6.

point V, not the centre; the opposite sides, if they meet, meet respectively in * one base; and it is the base in which those of such † inscribed one meet. If two opposite sides be parallel, the base is parallel to them.

Prop. XI. Fig. 12. If, in each of two circles having a common vertex to a given base, a chord be drawn through the vertex, and if, by joining the extremities of the two chords, a trapezium be formed having two points of concourse; the base is the connecting line. If two sides of the trapezium be parallel, they are also parallel to the base.

Thus: if, in BFT and ACK, BF and AC be chords through V the common vertex to the base LR, and if ABCF have two points of concourse; these are in LR. If FA and CB were parallel, they would be parallel to LR.

Dem. Through V draw NO parallel to LR. The opposite sides FA and CB, since they meet, cannot both be parallel to LR. Let FA, produced beyond A, meet LR in L, and cut NO in N. Through L and B draw LO, cutting NO in O; and draw LC, OC. Then, by the circle BFT, NV and VO are ‡ equal. And, because they are equal, then, by the circle ACK, AN (or FA) and OC produced will meet § in the base, therefore in L; whence OC and CL are one right line OL; of which, LB is part. Therefore LBC is one right line, and L a point of concourse. In like manner, some point R of LR is the other point of concourse. Therefore LR is the connecting line. And since, if FA or CB meet LR, they are thus proved to meet each other, therefore, if they do not meet each other, neither of them meets LR; that is, they are parallel to it. *Q. E. D.*

* Prop. I. † See Cor. 6. ‡ Cor. 2 to prop. VI. § Ib; converse.

Cor. 1. If the two circles coincide, ABCF becomes an inscribed trapezium: and thus is obtained a separate proof of the tenth proposition. *

Cor. 2. If, in any number † of circles having a common vertex and base, chords through the vertex be made diagonals of any trapeziums, inscribed or not; such base is the locus of all the points of concourse.

Prop. XII. Fig. 12. If a non-inscribed ‡ trapezium have two points of concourse; the connecting line is the base to which the mutual intersection of the diagonals is vertex in two circles, in each of which one diagonal is a chord.

Thus: if ABCF have L and R its points of concourse; LR is the base to which V, the intersection of AC with BF, is vertex in the circles ACK and BFT.

Dem. Let VD be the perpendicular on LR. Let ABC be the near angle, AFC the remote, and A between L and F. Through V draw NO parallel to LR, and indefinite. And let AC be first supposed to cut NO, and C to be nearer to LR than A is. Let lines bisecting BF and AC perpendicularly cut DV in E and G. With the centres E and G describe the circles BFT and ACK. In both, V is vertex to the base LR. For two circles, through F and A respectively, can § be described, each having V its vertex to LR. If these do not pass through B and C respectively, let them cut FB and AC in other points, X and Y; and complete the trapezium AXYF.

* The former proof is retained, as independent of harmonical proportion (involved in this proposition), and of the deduction by the coincidence.

† See Cor. 4 to prop. I.; or Cor. 5 to prop. VI.

‡ That is, one which cannot stand inscribed in a circle.

§ Cor. 5 to prop. VI.

Then FX and AY are chords through V in two circles having V their common vertex to the base LR; whence YX produced will * meet FA in L. And FY and AX cut NO, therefore are not parallel to LR, therefore † not to each other, but meet ‡ in a point of LR. This they cannot do, unless X be at B and Y at C: for, since X and Y are such that YX is directed to L, FY and AX will, if not passing through C and B, either meet before they cut LR, or beyond it. Therefore no two circles, passing through F and A respectively, except BFT and ACK, have V their common vertex to LR. Therefore those two have.

Let now AC be conceived § to coincide with NO. Then a circle, having its centre any point in DV indefinite, might pass through A and C. If a centre be found, in DV, by drawing DA or DC, and, perpendicular to it, a line from A or C cutting DV, then the circle will have V its vertex to LR; because DA and DC will be tangents.

Therefore, whether AC cut NO or not, LR is the base to which V is vertex in the circles BFT, ACK. *Q. E. D.*

Cor. 1. If FA and CB were parallel; RL parallel to them would be the base to which V would be vertex, in two such circles. For, let VD be a perpendicular on LR; and E, G, X, Y, as before. Then, if YX could meet FA, it would be || in RL. But RL does not meet FA. Therefore YX does not; but is parallel to FA and CB; whence AX and FY, which must ¶ meet in LR, must pass through B and C, and therefore meet in R; whence R is a point of the base to which V

* Prop. XI. † Ib; latter part. ‡ Prop. XI.

§ This may be conceived without an additional figure.

|| Prop. XI. ¶ Ib.

is vertex both in BFT and ACK. And this base, being * perpendicular to the right line through E, G, and V, is RL.

Cor. 2. If ABCF be a trapezium; one of those circles, as BFT, unless it coincide with ACK, includes it, and without contact. †

Cor. 3. If ABCF be a trapezium, either inscribed or not, having two points of concourse, L and R; the part NO, of the parallel to LR through V, intercepted between two opposite sides, is ‡ bisected in V. So between AB and FC.

Cor. 4. Fig. 13; the same as fig. 12, so far as the letters are common. If ABCF be a trapezium, inscribed or not, having two points of concourse L and R, and if FB and AC be the diagonals and V their mutual intersection, VD a perpendicular on LR, and right lines be drawn from D to F, B, A, C; the angles BDF and ADC are § bisected by DV. For the chord bisected in V is the tangent-chord of D, in each of the two circles of the proposition.

Cor. 5. Fig. 13. If FB and AC, produced, cut LR in I and K; then FI is to BI as FV to VB, and AK to CK as AV to VC. ||

Cor. 6. Fig. 13. The connecting line of a trapezium, inscribed or not, is ¶ divided harmonically in the four points L, I, R, K, in which it is cut by the produced sides and diagonals. For the demonstration, of the corollary here referred to, may (by the present proposition and its fifth corollary) be applied to trapeziums not inscribed.**

Cor. 7. Fig. 6. Hence, if L and R be the points of concourse

* Cor. 1 to prop. I. † Cor. 6 to prop. VI. ‡ Cor. 2 to prop. VI.
§ Cor. 3 to prop. I. || Cor. 1 to prop. VI. ¶ Cor. 3 to prop. X.

** This sixth corollary is proved differently in HAMILTON's Stereography, B. III. Sect. I. Lemma 22. The remainder of that lemma is the present Cor. 5.

of any conceived trapezium, whose diagonals produced cut the base in I between L and R, and in K beyond R, and if, on the diameters LR and IK, be described the semicircles LNR and ITK, and if IN and RT be perpendicular to LR; KN and LT, being drawn, are tangents.*

Cor. 8. Fig. 12 or 13. If the diagonals of a non-inscribed polygon, having an even number of sides, meet in one point; the opposite sides will respectively meet, if at all, in the base to which that point is vertex in two or more circles. For the proof applied † to the inscribed polygon may, now, be extended to the non-inscribed. If two opposite sides be parallel, the base is ‡ parallel to them.

End of the Introductory Propositions.

On the Elliptic Representation of a Circle, upon a plane surface, by Perspective.

Definitions.

Although the reader is presumed to be acquainted with the principles of perspective; yet the definitions in that science have varied so much, that it may be right to prefix a few here. Some new terms are also introduced.

Def. 1. The *centre of the picture* is that point, in the indefinite plane of the picture, in which it is cut by a perpendicular from the eye.

Def. 2. The *directing plane* is a plane through the eye, parallel to the picture.

Def. 3. The *intersecting line* and *directing line*, of an objec-

* Cor. 3 to prop. VI. † Cor. 6 to prop. X. ‡ Cor. 1 to this prop.

tive plane, are those in which that plane is cut by the picture and directing plane respectively: and the *centres* of those two lines are the points in which they are cut by perpendiculars from the eye.

Def. 4. The *intersecting point* and *directing point*, of an objective line or of its representation, are those two points in which the objective line cuts the intersecting and directing lines respectively.

Def. 5. The *director* is a right line from the eye to the directing point.

Def. 6. The *central line*, in an objective plane, is a right line cutting the directing line perpendicularly in its centre.

Def. 7. The *centroïd*, of an objective circle, is the point represented by the centre of the representing ellipse.

Def. 8. *Conjugate chords* are chords represented by conjugate diameters of the ellipse.

For brevity, the *representation* of any point A or line AB in any figure, is called *a* or *ab;* although the *a* or *ab* should not appear in the figure: small letters every where denoting points in the picture, whether exhibited in the figures or not. Each figure, after the first, represents a single plane: which, except in fig. 4 and 7, is the objective plane.

Prop. I. Fig. 1. The perspective representation of an entire circle, is an ellipse.*

Dem. Let the circle ABC be an object, viewed by an eye placed at O. Let FD be the plane of the picture, MR the directing plane, DE the intersecting line of the objective plane, and LR its directing line. Conceive an indefinite line, passing

* In these propositions, *ellipse* includes the case in which the representation is a circle; except the contrary be expressed or implied: a circle being an ellipse with equal axes.

always through O, to be moved round, along the circumference ABC. A cone will thus be generated. And the section of this cone by FD is the perspective representation of the circle. But, since the circle is supposed to lie wholly beyond FD, from O, the plane MR, through O, parallel to FD, neither cuts nor touches the circle. Therefore * the section is an ellipse. *Q. E. D.*

Cor. If the eye remain fixed, and FD be moved, parallel to itself, till it cut ABC; then a *part* only of ABC is seen through FD: and its representation is part of an ellipse, and similar to the corresponding part of *abc*. For a given figure will have similar projections, formed by the same lines OA, OB, &c, on parallel planes.†

Prop. II. Fig. 2. The centroïd of a circle is the vertex to the directing line taken as a base. ‡

Dem. Let GTH be the circle, LR its indefinite directing line, and V its vertex to the base LR. Then, whatever chord, as GH or ST, passes through V, its tangents meet § in a point of LR. Therefore they have the same director, and their representations are || parallel; whence those of GH and ST are ¶ diameters of the representing ellipse, and V is the centroïd. *Q. E. D.*

Cor. 1. Hence, from the directing line, the centroïd is found.**

* By the known properties of the cone. See, in particular, prop. 81, 82, 83, of Conic Sections by the Rev. T. NEWTON, Fellow of Jesus' College, Cambridge; 1794: which is the treatise referred to in subsequent notes.

† If the directing line touch or cut the circle, the representation is, accordingly, a parabola or hyperbola. It is not the design of these propositions, to enter upon those cases. They may be seen elsewhere: particularly in HAMILTON's Stereography; where also are various particulars of the elliptic representation.

‡ See Def. 2, of introductory Propositions. § Introd. prop. I.

|| BROOK TAYLOR's Perspective, 1719; Cor. 1 to Theo. V.

¶ By conic sections. A necessary converse of T. NEWTON's prop. 28.

** Cor. 1 to introd. prop. I.

Cor. 2. The centroïd remains fixed, if the circle and directing line remain: although the eye move, in the same directing plane; or the picture move parallel to itself; or vary its angle with the circle, the eye so moving as to preserve the same directing line.

Prop. III. Fig. 2. In conjugate chords, each is the tangent-chord of the directing point of the other.

Dem. Let GH and ST be conjugate chords. Produce them to their directing points R and L. Then, because *st* * and *gh* are conjugate diameters, *st* is † parallel to the tangents to the ellipse at *g* and *h*: whence the tangents to the circle at G and H have ‡ the same director, and the same directing point L, with ST; or GH is the tangent-chord of L the directing point of ST. In like manner ST is the tangent-chord of R the directing point of GH. *Q. E. D.*

Cor. 1. If GH, the tangent-chord of any point L in the directing line, will cut that line in any point R; L and R will be the directing points of conjugate chords. For ST the tangent-chord of R will § pass through L; whence the tangents to the ellipse at *g* and *h* are parallel to *st*. And those at *s* and *t* are parallel to *gh*. Therefore *st* and *gh* are conjugate diameters.

Cor. 2. If M be the || mean point, and L and R the directing points of conjugate chords, LMR is ¶ a right angle. And, if LR be the directing line and LMR a right angle, L and R will be the directing points of conjugate chords. For the tangent-chord of each point will pass ** through the other.

* See what is subjoined to the definitions. † By conic sections.

‡ A necessary converse of Cor. 1. to Theo. V of Brook Taylor.

§ Introd. prop. III. || Introd. prop^{s}. def. 3. ¶ Ib. prop. III.

** Ib. prop. IV.

Cor. 3. If therefore one chord through the centroïd V were the diameter, consequently * perpendicular to LR and passing through M; the conjugate chord would be parallel to LR, therefore bisected in V.

Prop. IV. If any right-lined figure, having an even number of sides, be inscribed in a circle, or described about one, and if its diagonals, joining opposite angles, pass through the centroïd; the opposite sides of the representation are parallel respectively.

Dem. If the circle be parallel to the picture, the centroïd is the centre, and all the respectively opposite sides, of the inscribed or circumscribed figure, are parallel; therefore also their representations. Let the centroïd be another point. The opposite sides † will meet, if at all, in the indefinite base to the centroïd taken as a vertex, that is, in ‡ the directing line of the objective plane. If any two opposite sides will not meet, they are § parallel to the directing line. Therefore, in all cases, the representations are parallel. *Q. E. D.*

Cor. This is applicable, *mutatis mutandis*, to non-inscribed figures.||

Prop. V. Fig. 3. If the centre of an objective circle be in the central line of its plane; one axis of the representing ellipse is parallel to the intersecting line, the other perpendicular.

Dem. Let V be the centroïd, A the centre of the directing line LR, and AO, in the central line, equal to the distance of A from the eye; BF the diameter through V, and PQ the chord bisected in V. Then, since FA cuts the intersecting line

* Cor. 1, 2, to introd. prop. I.

† Introd. prop. X. and cor. 6, 8, 9.

‡ Prop. II.

§ Ib. with cor. 1, 6, 8, 9, of introd. prop. X.

|| Introd. prop. XII, and cor. 1 and 8.

perpendicularly in its centre, therefore (by perspective) *fv* is perpendicular to that line, and *vq* parallel: whence *fvq* is a right angle. But no two conjugate diameters are at right angles, except the axes. Therefore *bf* and *pq*, always * conjugate diameters, are now the axes. *Q. E. D.*

Cor. 1. The perpendicular axis has (by perspective) its vanishing point in the centre of the vanishing line of the objective plane.

Cor. 2. It is also directed to the centre of the picture. For the directing line is † perpendicular to that plane, through the eye, which is perpendicular to the objective and directing planes. Therefore the central line is in that perpendicular plane: whence its indefinite representation is the common section of that plane with the picture; which common section passes through the centre of the picture.

Cor. 3. If O be at the mean point M (which is always ‡ in BF), in which case the distance of A from the eye is equal to a tangent from A; the representation is a circle, whose centre represents V. For, from the further extremity F of BF, draw FP; also FQ, produced to R in LR. Draw MR, OR, AQ. Then, if AOR, revolving on AR, bring O to the eye, AO and OR become the directors of *fv* and *fq*, therefore parallel to them respectively; whence the angle *vfq* is § equal to AOR, and the triangle *vfq* similar to AOR, wheresoever O is in AF. But, if O be at M, AO is equal to AR. For the angle AQR is equal || to FPQ, to FQP, to ARQ; whence AR is equal to

* Cor. 3 to prop. III.

† XI Eucl. prop. 19.

‡ Cor. 2 to introd. prop. I.

§ XI Eucl. prop. 10.

|| Its vertically opposite is so. III Eucl. prop. 32.

AQ, so to AM; therefore *vq* to *vf*, and the axes are equal: but V remains the centroïd.

Cor. 4. If AO and AM be unequal, the axes are so. If AO be greater than AM, the axis perpendicular to the intersecting line is the major axis; if less, the minor.

Cor. 5. Fig. 4. If O be at the mean point, the section of the cone of rays is the subcontrary * section. For, let ABF be as before, but O now at the real place of the eye; so that fig. 4 shall be the plane in which are BF and its representation *bf*. Draw AO, OF. Then *bf* is parallel to its director AO. And, since AO is equal to a tangent from A, its square is equal to the rectangle FAB; whence the triangle FAO is similar to OAB, and the angle OFA or OFB equal to BOA, so to O*bf*.

Cor. 6. Fig. 4. If OFB and O*bf* be unequal, the axes are so. If OFB be greater than O*bf*, the axis perpendicular to the intersecting line is the major axis; if less, the minor.† For, if A and B and F be fixed, whilst, by the motion of the eye directly from or towards A, the angle OFB and the length AO are varied; these will increase or decrease together, and O*bf* will accordingly decrease or increase.

Prop. VI. Prob. I. Fig. 3 and 5. A circular object being given in magnitude, and in position with respect to the eye and the plane of the picture; to find the axes of the representing ellipse.

Fig. 3. If the centre of the circle be in the central line of the plane, find (by perspective) *bf*, *pq*; which are ‡ the axes.

Fig. 5. If it be not so placed, let the circle GTH be the object, and E its centre. From the given position the

* Known in conic sections. See T. NEWTON, prop. 82.

† Cor. 4.

‡ Prop. V. and its *Dem.*

directing plane is determined, and the directing line. Let this line be LR, and A its centre. Make a right angle LAO, on the same side of LA as the circle is; and make AO equal to the distance of A from the eye. Find V the * vertex to the base LR; also M † the mean point. Draw OM; and bisect it perpendicularly in Q, by QC cutting LR in C: and draw CO. In LR take CL and CR, each equal to CO; and, through V, draw, towards R and L, the chords GH and ST. Find *gh* and *st*, by perspective. These are the axes.

Dem. With the centre C and radius CO, describe the semicircle LOMR; and draw LO, OR, LM, MR. Then, because V is ‡ the centroïd and LMR a right angle, *gh* and *st* are § conjugate diameters. And they are perpendicular to each other. For, if LOR, revolving on LR, bring O to the eye, OL and OR become the directors of *st* and *gh*, therefore parallel to them respectively; whence the angle *tvh* is || equal to the right angle LOR. But no two conjugate diameters are perpendicular to each other, except the axes. Therefore *gh* and *st* are the axes. *Q. E. D.*

In *practice*, find (by trial) in the directing line a point C equidistant from O and M; and, in the same line, take CL, CR, each equal to CO. And, if either L or R, suppose R, be at an inconvenient distance, the chord GH, directed to R, may be found as the tangent-chord of L. ¶

Cor. 1. Fig. 6; the same as Fig. 5, so far as the letters are common. One mode of finding the axes, after GH and ST found, is this. Draw GT, and produce it cutting LR in B. Draw OB; also BK, cutting OR perpendicularly in K. Find,

* Cor. 1 to introd. prop. I. † Cor. 2. ib. ‡ Prop. II.
§ Cor. 2 to prop. III. || XI Eucl. prop. 10. ¶ Prop. III.

by perspective, *gt*, or *gv*, or *vt*; and, upon the line so found, make a triangle *gvt* similar to OKB, having the angle *g* equal to O, and *v* to K. Then are *gv* and *vt* two semi-axes. For R and B are the directing points of GV and GT.

Cor. 2. Fig. 5, 6. When E is not in the central line of the plane, neither GH nor ST is perpendicular to LR; therefore neither axis tends to the centre of the vanishing line of the plane.

Cor. 3. Fig. 5, 6. Wheresoever O is in the arc LMR, the same chords GH and ST are the originals of the axes.

Cor. 4. Fig. 6. The major axis has its directing point on that side, of the central line AO, on which E is placed. For, draw ML, MB, MR. Then, if O were at M, *gv* would be * equal to *vt*, and the angle *vtg* to *tgv*. Therefore LMB is † equal to BMR. But, if O be in the arc between L and M, the angle LOB stands on a greater arc in the circle LMR than LMB does, and BOR on a less than BMR; whence LOB, therefore OBK, is greater than BOK, OK than KB, and *gv* than *vt*.

Prop. VII. Fig. 7. If the object be a sphere in any oblique ‡ position; the major axis, of the representing ellipse, is directed to the centre of the picture.

Dem. To a given sphere conceive any tangents drawn from O the eye. The points of contact are in the circumference of a circle, whose representation (as to the outline) is that of the

* Cor. 3 to prop. V.

† Or, without perspective, these angles are equal; by Cor. 5 to introd. prop. X: where they are (fig. 10) lMR, RMK.

‡ By *oblique* position is here meant any except that in which a right line, from the eye to the centre of the sphere, passes through the centre of the picture.

sphere, and whose plane is perpendicular to a line * from O through E the centre of that circle to the centre of the sphere. If this line passed through the centre of the picture, such plane would be parallel to the picture, and the representation a circle. But, let it not so pass. Yet such plane is to be taken as the objective plane. And, since E is in every plane perpendicular to it through the eye, and the central † line is in one such plane, and both are in the objective plane, E is in the central line. Let FB, the diameter in that line, have B its extremity nearer to its directing point A, and be produced to A. Draw OF, OB, OA; and let the representation *fb* meet FB produced in K. Then the angle OFB, being equal to OBF, is ‡ greater than B*b*K or O*bf*. Therefore the axes are unequal, and the major axis is directed to the centre of the picture. § *Q. E. D.*

Cor. 1. If the plane of a picture parallel to the former should cut off part of FB; then the part viewed, of the circle or sphere, would have a representation || similar to its former one, and the major axis would still be directed to the centre of the picture.

Cor. 2. The centre of the picture is not the vanishing point of the major axis. For FAO or FK*f* is not a right angle.

Lemma. Fig. 8. No. 1 and 2. If original lines, in one plane, converge to a point; their representations will converge to a point, or be parallel. And, if the representations converge to a point; the originals will so converge, or be parallel.

* The intersection of this line with the picture is also its vanishing point. When this is given, the vanishing line of the perpendicular plane is found by perspective. Brook Taylor, Prob. XV.

† In Cor. 2 to prop. V.

‡ I Eucl. prop. 16.

§ Cor. 6 and 2 to prop. V.

|| Cor. to prop. I.

Dem. Let the indefinite original lines CF, DG, EH, meet in P; having C, D, E, their directing points, and F, G, H, their intersecting points. From A the centre of the directing line, and towards FH, draw AO perpendicular to AD and equal to the distance of A from the eye. Draw CO, DO, EO: which are the directors, brought down, into the original or objective plane, by the revolution of COE or the directing plane on CE. Parallel to these draw lines through F, G, H: which will be the indefinite representations, brought down by the revolution of the picture on FH. These indefinite representations will form, on FG and GH, triangles respectively similar to COD and DOE; and perpendiculars from their vertices to FH will bear to OA the ratios of FG to CD and GH to DE. But these ratios are equal, because FH is parallel to CE. Therefore the perpendiculars are the same line KN, and the vertices are one point K. If P be in the directing line, by which C, D, and E, coincide with P; there is only one director, and the representations are * parallel. One part, then, is proved.

Now, let F, G, H, be the intersecting points, and FK, GK, HK, the indefinite representations, brought down as before: and let KN be a perpendicular on FH, and let AO be as before. Draw OC, OD, OE, parallel respectively to KF, KG, KH, and cutting the directing line in C, D, E; and draw, indefinitely, CF, DG, EH. These are the original lines represented; because C, D, E, are the directing points, and F, G, H, the intersecting points. From similar triangles, CD is to FG as CO to FK, as AO to NK, as EO to HK, as DE to GH; or CD is to FG as DE to GH. Therefore, if CD and FG be unequal, consequently DE and GH, then CF, DG, EH, will

* BROOK TAYLOR; Cor. 1 to Theo. V.

meet in a point P, of DG, making PD to PG as CD to FG. And, if CD and FG be equal, the three will either meet in a point equidistant from FH and CE, or be parallel. *Q. E. D.*

Cor. 1. If the originals neither converge to a point nor are parallel, the representations do not converge to a point.

Cor. 2. It is impossible, therefore, that the major or minor axes should so converge, of ellipses representing any circles; unless the originals of the axes should in any case so converge or be parallel: which will only be, if the intervals between their intersecting points be proportional to those between the directing points.*

Prop. VIII. Prob. II. Fig. 9. The magnitude and positions being given, of any equal circles in one plane, having their centres in the central line; to find a law which the axes, of the representing ellipses, observe in their directions.

One axis is † parallel to the intersecting line, the other perpendicular. Let AR, KI, AK, be the directing, intersecting, and central lines, indefinite; AK passing through the centres of the circles *i*, *ii*, *iii*, and any others such as supposed. Conceive O (not in the figure) at the real place of the eye. AO may be equal to a tangent from A to some one circle, or not. Let the *ii* be so placed. This alone will be ‡ represented by a circle: those § between KI and *ii* by ellipses whose minor axes are parallel to KI, and major axes directed to the centre of the picture; those beyond *ii* by ellipses whose major axes are parallel to KI, and minor axes directed to the centre of the picture. This, then, is a *law* observed.

* Yet some advantage may be gained, by discovering any *law* or *regularity* observed in the directions of the axes. This is attempted in the remaining propositions.

† Prop. V. ‡ Cor. 3 and 4 to prop. V. § Cor. 4 and 2 to prop V.

Cor. 1. The law holds in unequal circles, when all nearer and further than *ii* have, respectively, their mean points nearer and further than that of *ii*: because the distance of the mean point from A is equal to the tangent. But the inequality may be such as to destroy this supposition.

Cor. 2. If AO be less than AK, therefore less than the parallel and equal line (OS in Fig. 4) which is the distance of the vanishing line from the eye*, and if the mean point of the nearest circle be beyond the picture; then all the major axes are parallel to KI, all the minor directed to the centre of the picture.

Cor. 3. When it is known which axis is parallel to KI, then, if one point of either be given, its direction is got by drawing a line to the centre of the picture, or else a line parallel to KI.

Cor. 4. The extremities of the axes parallel to KI, are bounded by the following limits. Let PQ be the tangent-chord of A in the nearest circle, and GH the parallel diameter; G and P being on one side of AK. If lines be drawn through G and P, parallel to AK, their representations are the boundaries of the extremities on that side of AK; and the like on the other side. For PQ is the original of the axis: and the corresponding chord is greater, as the circle is more distant; but always less than the diameter.

Prop. IX. Prob. III. Fig. 10. To find the same, when the centres are in a line parallel to the central line.

Let one rank of equal circles so placed be *i*, *ii*, *iii*, &c.; let

* This supposition may, when the objective plane is perpendicular to the picture, be expressed thus: If the height of the eye be less than the distance of the picture. Where *height* denotes perpendicular distance from any objective plane.

another be *ix, x, xi*, &c.: all having their centres E, E, &c. From the directing line AN, find the centroïds V, and the mean points M, in as many circles as may be requisite. On the same side of AN as the circles are, take AO, in the central line, equal to the distance of A from the eye; and make A the centre of a circle ON. In either rank, as *i, ii*, &c, if all the mean points be without ON, find * the directing point L of the minor axis of *i* the nearest circle. By the construction, the directing points of the minor axes for all the circles *ii, iii*, &c, will be between L and A. Draw LO. In the plane of the picture place the triangle LOA, or any similar triangle, in the same situation with regard to the indefinite intersecting line, or some parallel line, as LOA has here with regard to the directing line. Then † will all the minor axes, for *ii, iii*, &c, be respectively parallel to lines drawn from points in the base of such triangle to its vertex corresponding with O; each of such points will be nearer to the point corresponding with A, as the circle is further from the directing line; and the approach to A (and to its corresponding point) will be without limit, while circles are added. This, then, is a *law* observed by the minor axes: to which the major will be perpendicular, respectively.

If the circles be unequal, the law holds good; unless the inequality cause the mean point of a further circle to be not further than that of a nearer circle.

If the angle LOA be inconveniently large, find also the directing point F for the circle *ii*. Then all the directing points, for *iii*, &c, will be between F and A. Or, in some cases, find the directing point G for the furthest of the rank: and then the

* Prop. VI. † Ib.

directing points, for all between i and the furthest, will be between L and G.

The use of the law found, is this. When, in practice, it would be tedious to find * accurately, in position and length, the axes for each circle; the artist may, by this law, know pretty nearly the directions of the axes for all the circles, of the rank for which he has constructed the triangle LOA or FOA. Those axes, howsoever seldom they may converge † to a point, are always parallel to lines which do. In addition, he may find, by perspective, such particular points as any case may suggest. If ST be the diameter perpendicular to the directing line, st is readily found: which passing through v, so being a diameter of the ellipse, its middle point is one point in each axis. Or, some centroïds V may be found; whence the elliptic centres v, by easy operations of perspective. If the circles are equal, it may suffice, after V found in a few of the nearest circles, to take the remaining lines EV (in the same rank) equal, or to diminish them by the view. And, if the circles (in each rank) be also at equal intervals, their centroïds may often be considered as at equal intervals; or, after a few of the nearest. Whence the points v, with increased facility.

If the mean point in any circle, as ix, be within ON, find the directing point R of the major axis: which may be more convenient than that of the minor, as nearer ‡ to A. But this case cannot happen, with the mean point lying beyond a picture perpendicular to the objective plane; unless the perpendicular distance (by the scale assumed for any delineation) of

* By prop. VI. † See Cor. 2 to Lemma.

‡ If M be in the circumference ON, the directing points of the two axes are equidistant from A; by the construction of prop. VI.

that plane from the eye be greater than the distance of the picture.

If a figure, constructed as fig. 10, happen to be inconveniently large, it may be constructed upon a reduced scale. The triangles LOA &c. remain similar, each to itself; whence the directions of the axes are the same.

Prop. X. Prob. IV. Fig. 11. To find the same, when the centres are in a line parallel to the picture.

Of the circles *i*, *ii*, &c, let the centres be E, E, &c.; let AO and ST be as in Prop. IX, and D, D, the directing points of ST. In one circle, find the centroïd V and the mean point M. Through these draw lines parallel to AD: which will cut the other lines ST in the points V and M of all. Let the parallel through M cut AO, or AO produced, in F.

First case. Let AO be less * than AF. Let *ii* be so placed, that OD, being drawn, may equal DM or AF. Find G the directing point of the minor axis for *ii*; and draw GO. Transfer † GOA to the picture; (or place therein a similar triangle.) Then the minor axes, for all the circles on the same side of AO as *ii* is, are parallel to lines from points of GA to O; and, the nearer any such circle is to *ii*, on either side of it, the nearer will the axis be to parallelism with GO. Nor is there any limit to the approach to parallelism with GO, or, reversely, with AO. This, then, is a law observed.

For the circle OM ‡ has the shortest radius when its centre is at the D of *ii*. That radius is increased, whether M (always

* This is always so, when the picture is perpendicular to the objective plane, if (by the scale) the perpendicular distance of the plane from the eye be less than the distance of the picture; unless a mean point be between the picture and AD.

† As LOA in prop. IX.

‡ Drawn as in prop. VI.

in the parallel through F) be taken between F and the M of *ii*, or beyond that M; and the circumference cuts GA between G and A, approaching to A as the radius is increased. And both the increase of the radius and the approach of the circumference to AO are without limit, while circles are added.

The *i* and *iv* are so placed, in the figure, as to have the same directing point for their minor axes: C being the centre of OMM.

Second case. Let AO be AF. Produce AG to H, making AH equal to AF. Then, the nearer M is to F, the less is the radius of the circle OM or FM; never less than AF. At the same time, its centre is the nearer to A, and the directing point of the minor axis is the nearer to H: which approaches may be made without limit. Place, as before, in the plane of the picture, a triangle; now HFA, or one similar. The minor axes will be parallel to lines from the base of such triangle to its vertex: each, as its circle is further, being nearer to a right angle with the intersecting line; and without limit. This is a law observed.

If requisite, find accurately the directing points of the minor axes for a few circles nearest to AF: or for one; as N for *iv*. In HFA, transferred to the picture, mark N. Then, for all the circles nearer than *iv*, the minor axes will be parallel to lines from points between H and N; and, for all beyond *iv*, between N and A.

In this second case, if M be at F, the representation is * a circle.

Third case. Let AO be greater than AF. With the centre A describe a circle LOR; cutting AD in R on the same side

* Cor. 3 to prop. V.

of AO with the objective circles before given, and in L on the other side. Then,* if one of those circles, as *ii*, have its mean point in the circumference OR, the directing points of the two axes are L and R. And, for any circle between *ii* and AO, the directing point of the major axis is between R and A, and nearer to A than that of the minor is; which is beyond L. And, for all circles beyond *ii*, the directing points of the minor axes are between L and A. Draw LO, RO. Then LOA, ROA, or triangles similar to them, transferred to the picture, as † GOA in the first case and HOA or HFA in the second, give lines to which the axes will be respectively parallel: and, of the minor axes, each, as its circle is further, is nearer to a right angle with the intersecting line; and without limit. This is a law observed.

Any requisite approximation to accuracy may be made; by the mode used in the second case, when N was found.

A figure, constructed as fig. 11, may, if required, be reduced by a scale.‡

Prop. XI, and last. Prob. V. Fig. 12, 13. To find the same, when the centres are in a right line oblique to the intersecting line of the plane.

Fig. 12. If the circles had been so placed that (instead of the centres) the mean points, M, M, &c, were in such line having any directing point X; a law observed would have been as follows. Let LR be the directing line; and let AO be as before, but less than AF, and in F let AO produced meet XM. With the centre A describe a circle, having the diameters ON, LR; L being on the same side of A with X: and let XM

* Construction of prop. VI.

† As LOA in prop. IX.

‡ See the end of prop. IX.

be wholly out of LOR. In FM take FP towards X, and FQ the contrary way, each a mean proportional between FN and FO; and draw PC and QK, perpendicular to PQ, cutting the directing line in C and K. Let the circle PG with the radius CP cut LR in G towards R, and let the circle QH with the radius KQ cut it in H towards L. Let the M nearest to X be between X and P. In the directing line find the centre W of a circle passing through O and this M, cutting FM in U also, and LR in I beyond A. Then, for circles whose mean points are U and the M nearest to X, the directing point of the minor axis is I. If a circle move, and so that its mean point shall move in XM; then, while this moves from that M to P, and on from P to U, the directing point of the minor axis moves accordingly from I to G, and back from G to I. While the mean point moves from U to F, from F to Q, then beyond Q without limit, such directing point, accordingly, moves from I to A, then from A to H, then approaches to A without limit. From these directing points triangles may be constructed*, and transferred to the plane of the picture; giving lines parallel, as before, to the minor axes. Such, then, would have been a law observed, if the mean points had been in a right line.

For, describe the circles PON, QON. Their centres are † C and K. Therefore they are the circles PG and QH; that is, the circles OM ‡ with mean points at P and Q. Now, since all the circles OM have§ their centres in the directing line, and since, if a centre were assumed between C and K, the circumference through O could not meet XM, no centre is between

* As LOA, GOA, &c; in prop. IX, X.

† III EUCL. prop. 1, 37, 19.

‡ According to the construction of prop. VI.

§ Ib.

C and K; and CP is the shortest radius, and, when M is beyond F, KQ is the shortest; and, the further any centre W is beyond C, the more does its circumference fall within the circle POGN at OIN, and the more without it elsewhere. And the like beyond K. Therefore, as the mean point moves from the first M to P, the circles OM become smaller, and cut LR in points further beyond A; which are the directing points of the minor axes; and the reverse takes place, while the mean point moves from P to U, and so on to F. As it approaches to F, the radius of OM increases without limit. When it is at F, the arc OM becomes a right line; which passes through A: whence the minor axis is * perpendicular to the intersecting line. While the mean point moves from F to Q, then beyond Q, the radius of OM decreases to KQ, then increases without limit; and the intersection of the circumference OM with LR moves from A to H, then approaches to A without limit.

The cases of AO equal to AF and greater, bear a resemblance to the second and third cases of Prop. X; and are not perhaps sufficiently frequent, in practice, to justify an examination here. And, whatever be the length of AO, if any mean point fall within LOR, it is to be remembered † that the directing point of the major axis is then nearer to A than is that of the minor.

Fig. 13. Now let the centres E, E, of the objective circles, be in a right line. Let LR and AO be as before; also the diameters ‡ ST perpendicular to the directing line; D, D, being their directing points, B that of the line through the

* As also by Cor. 4 to prop. V.

† Prop. IX, last paragraph but one; and X, 3d case.

‡ As in prop. IX, X.

centres, and H that of the line through the points T, the nearer extremities of ST. Let one of the circles have its centre in the central line; let F be its mean point, and AO less than AF. Let the circles *i*, *ii*, and any others, be on the same side of AO with B; and *ix* &c, on the other. Take any one, as circle *ii*, and draw TH; also HN, parallel to DT, cutting BE in N; and produce HB to Q, making BQ equal to HB. Then DT is to DH as DE to DB, as HN or TE to HB, as ES to BQ; whence DT is to DH as DS to DQ, and the rectangle SDT or the square of DM to QDH as the square of DT to that of DH, therefore as that of HN to that of HB. This is * a property of the hyperbola. Therefore all the points M are in the curve of an hyperbola, whose transverse axis is QH and conjugate axis equal to twice HN, that is, to a diameter ST; having the vertex H, and the † asymptote BE.

What follows is an attempt to make such approximation, to some law or regularity observed in the directions of the axes, as may be of use without constructing the hyperbolic curve.‡

As to the remoter circles, it may often be sufficiently accurate, in practice, to consider their mean points as in a right line; either drawn through one of them, parallel to BE, or drawn from F to the mean point of the furthest. So far as this can be admitted, the theory already laid down, respecting mean points in a right line, is to be taken also as the law or laws observed where the centres are so placed. If, in any case, one such line be not sufficiently accurate for all the circles

* T. Newton, prop. VI. † Ib. prop. XXII.

‡ The construction of prop. VI is to be kept in mind: referred to already in prop. IX, X, XI.

beyond F (from B), two such lines may be assumed, at different distances from BE; the one nearer to BE being for the remotest circles. As to mean points between F and H, yet not within the circle LOR with the centre A; find the minor axes, each separately, for as many, of the nearest to H, as requisite. If any remaining circles on that side of AF, as *ii* &c, should seem better referred to some law observed; then, with the centre W equidistant from O and the M of *ii*, describe a circle MOI cutting the directing line in I beyond A. If any mean point be within this circle, the directing point of the minor axis is beyond I. And, if any other objective circle have its mean point U in the circumference MOI, or if U in MOI can be found as the mean point of a circle purposely added or supposed (of the given magnitude, and having its centre in BE), then any mean points between that M and U will have the directing points of the minor axes beyond I. If how much beyond be inquired; the points P and G may be found, as (fig. 12) when the mean points were in a right line: and G will be a limit beyond which, at least, the directing points cannot go. And U will perhaps frequently be found with sufficient accuracy, if taken as the intersection of the right line MF with MOI. If the mean point come to U, the directing point returns to I; and if the mean point move from U to F, the directing point moves from I to A. Triangles constructed as * before, and transferred to the picture, will give lines which may be taken as parallel to the axes; though with some abatement from strict accuracy. But such is a law, or laws, which may be considered as observed, when the centres are in a right line.

* As LOA, GOA, &c; in prop. IX, X.

As to the cases of AO being AF and greater, and that of mean points within the circle LOR with the centre A; it seems unnecessary to add to what was said in concluding that part, of this proposition, in which the mean points were supposed to be in a right line.

Since the axes of the hyperbola are given in length and position, it could be described. And its intersections with the diameters ST would give all the mean points. Whether this would aid the perspective operations so much as to compensate for the labour of constructing the curve, I leave to others to examine.

RICHARD HEY.

March 1814. Hertingfordbury, near Hertford.

Fig. 1.

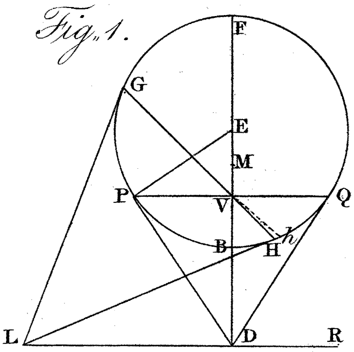


Fig. 5.

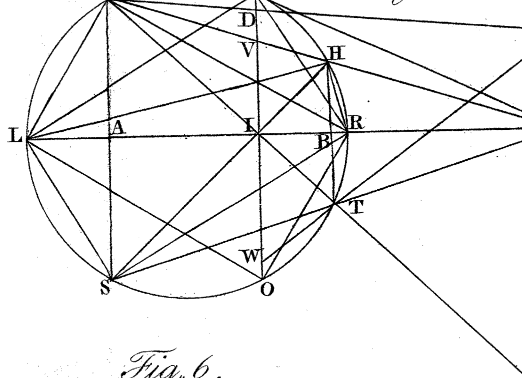


Fig. 2.

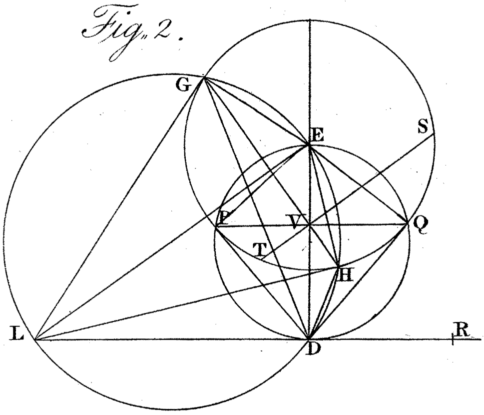
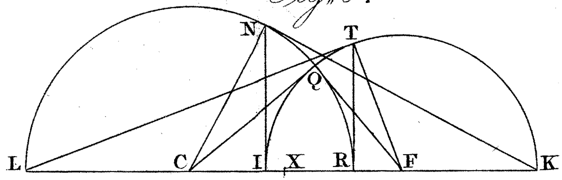


Fig. 6.



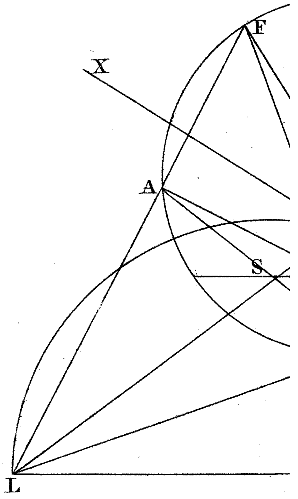


Fig. 3.

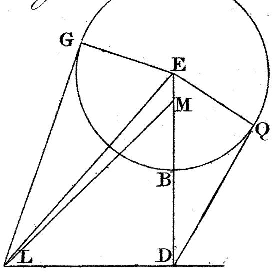


Fig. 7.

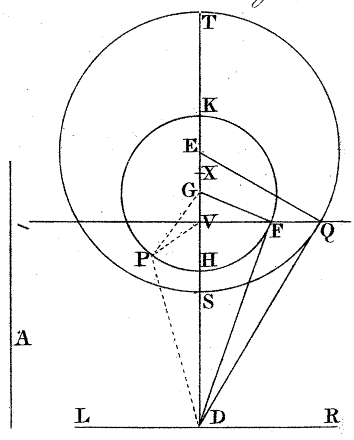


Fig. 4.

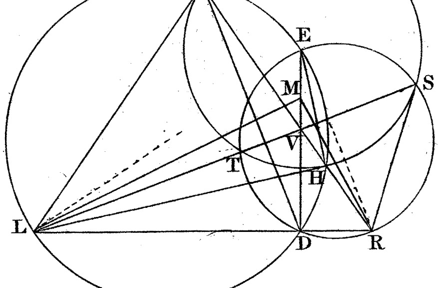
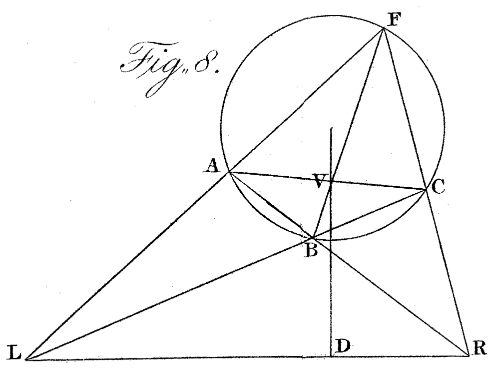
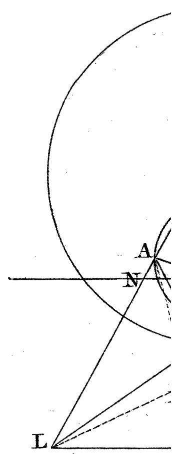


Fig. 8.





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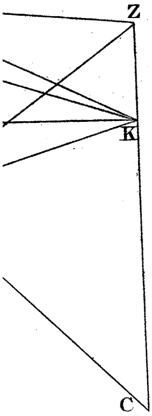


Fig. 9.

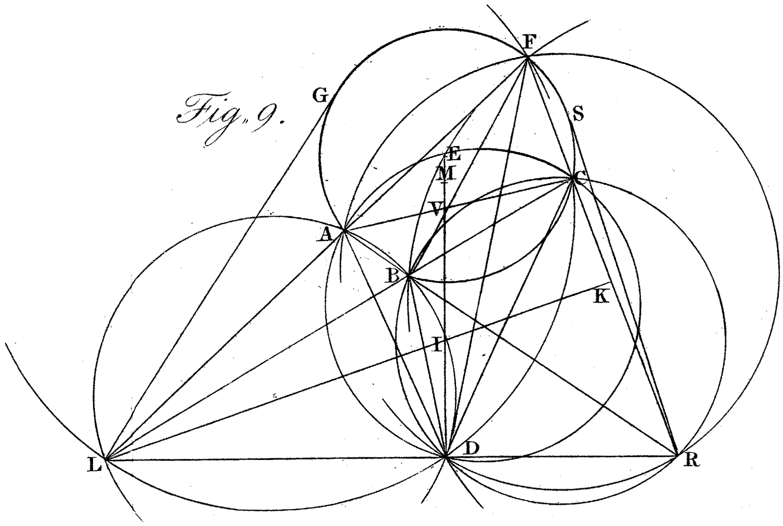


Fig. 10.

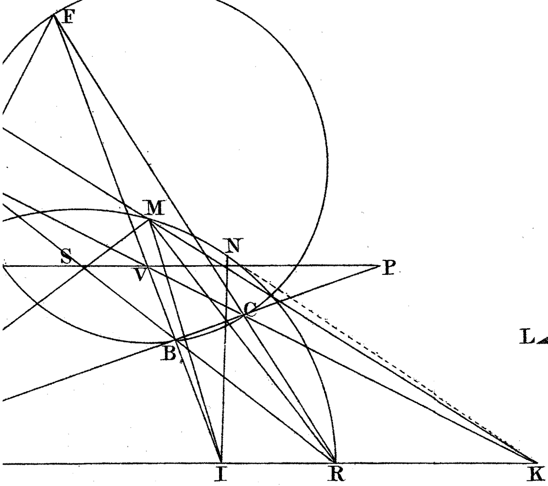
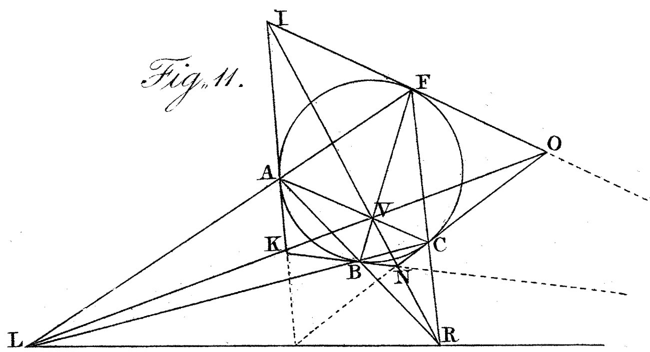


Fig. 11.



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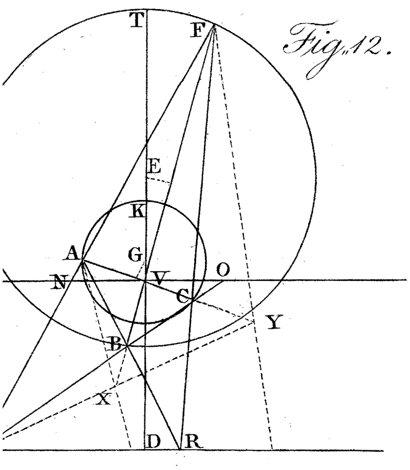
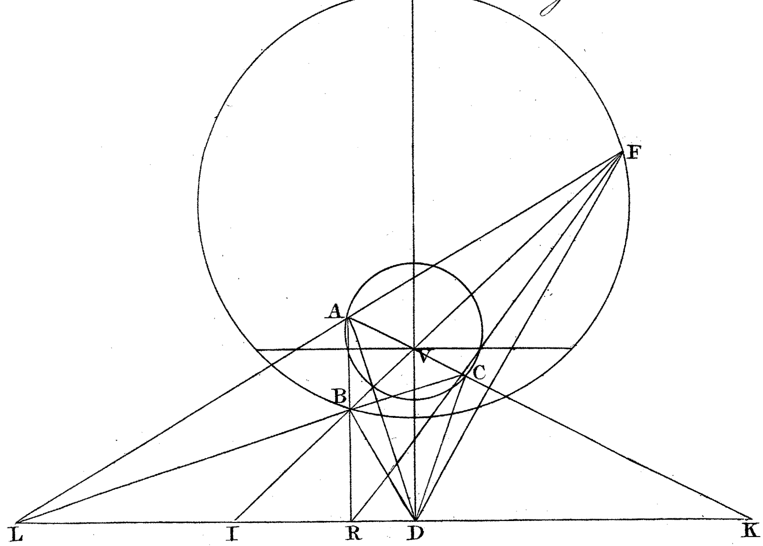
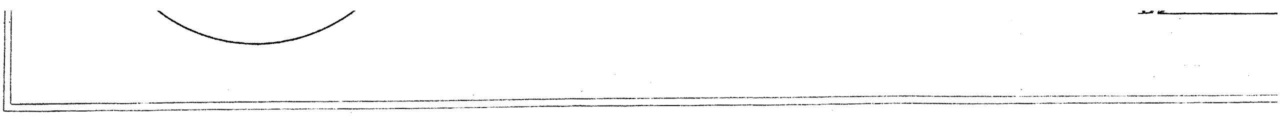
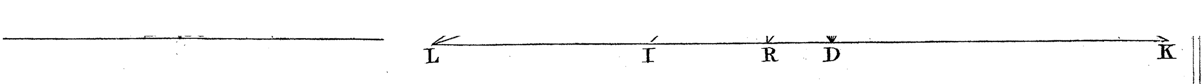


Fig. 13.







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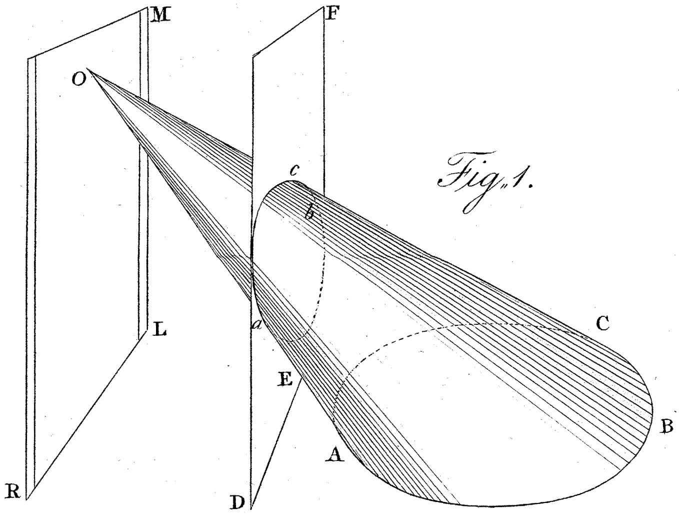


Fig. 1.

Fig. 2.

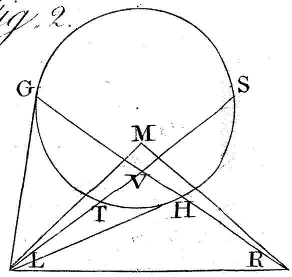
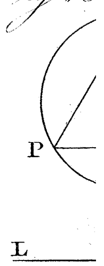


Fig. 3



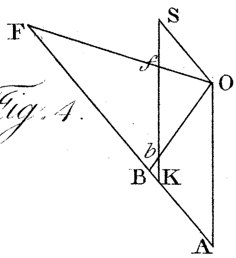
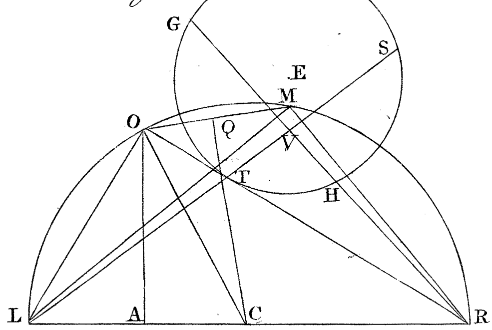
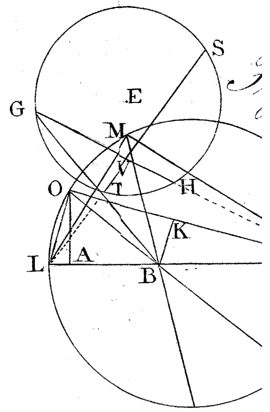


Fig. 4.

Fig. 5.





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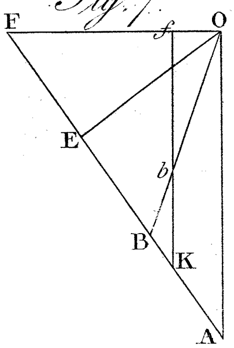
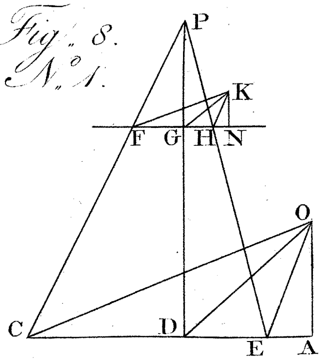


Fig. 8. N.o 1.



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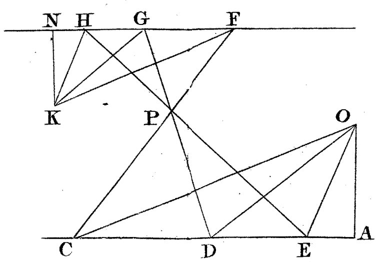

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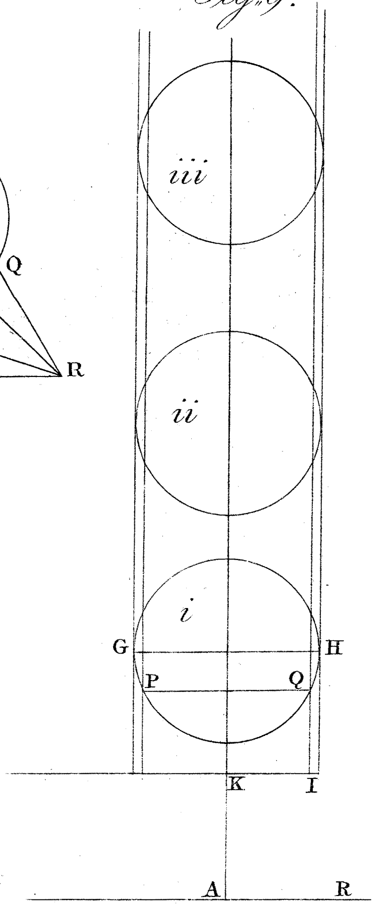

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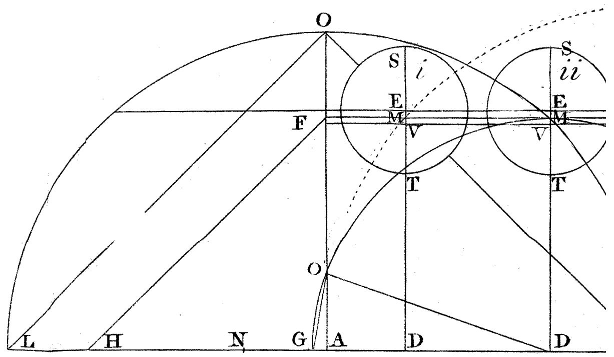

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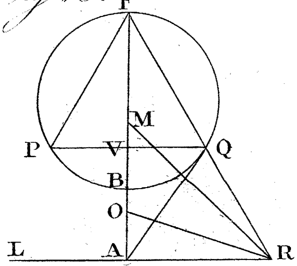

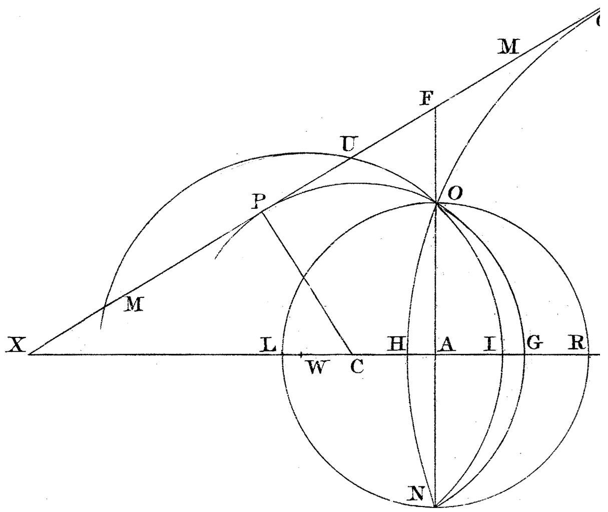

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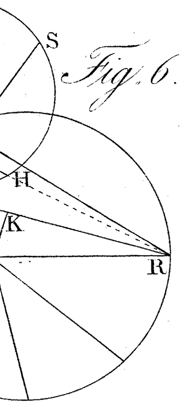

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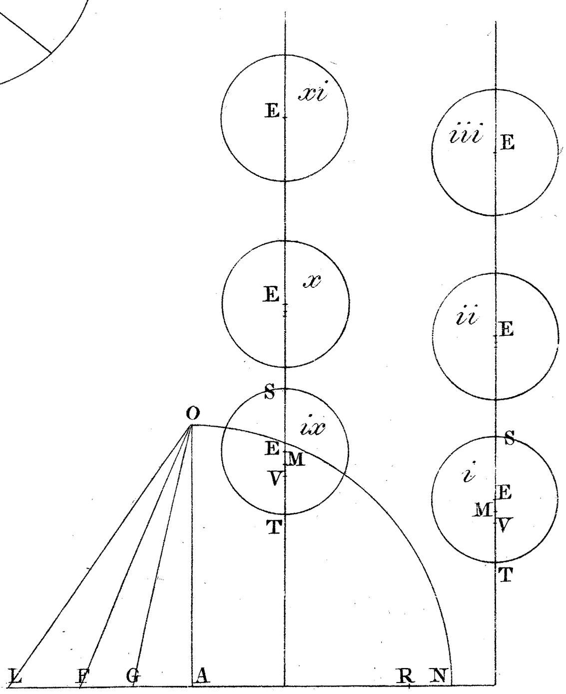

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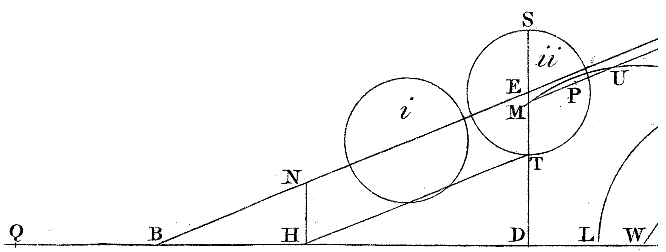

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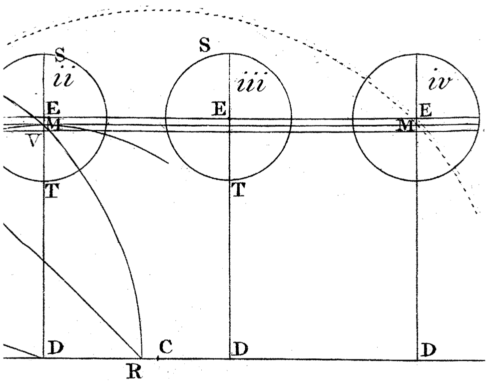


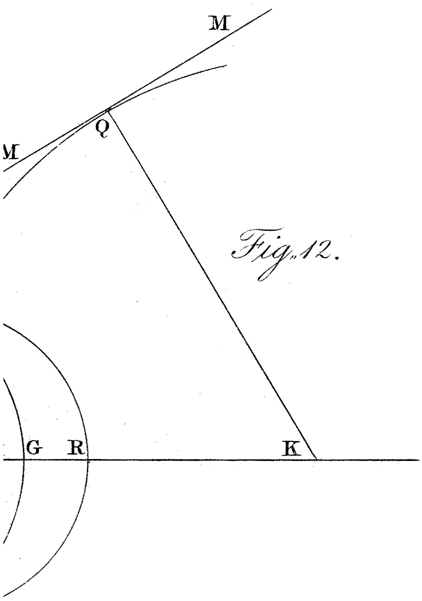
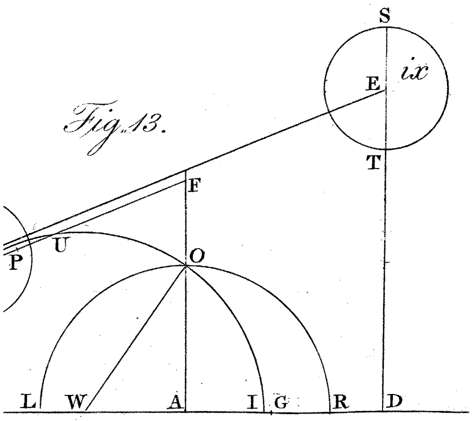


Fig. 12.

Fig. 13.



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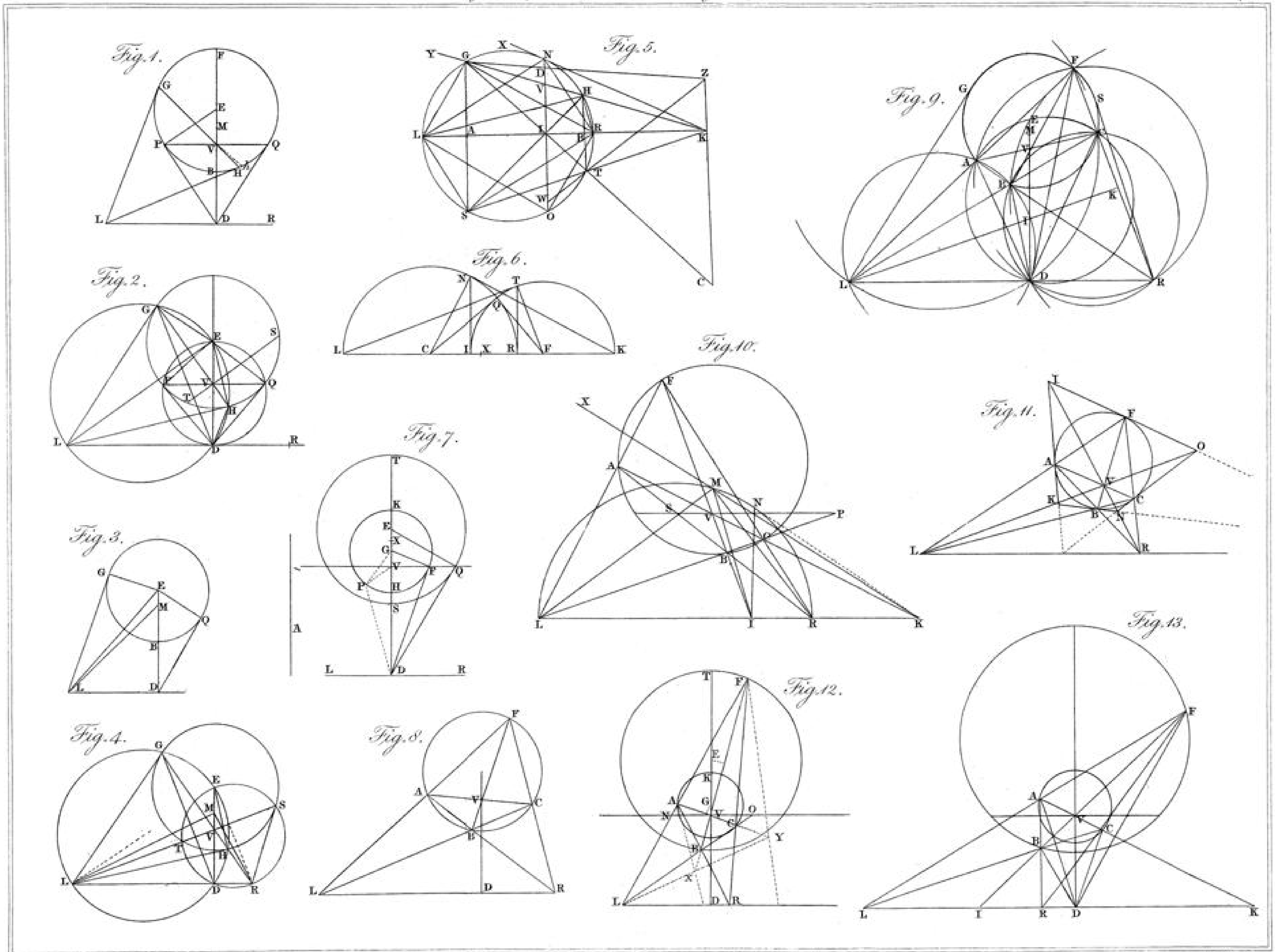
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.
Fig. 12.
Fig. 13.

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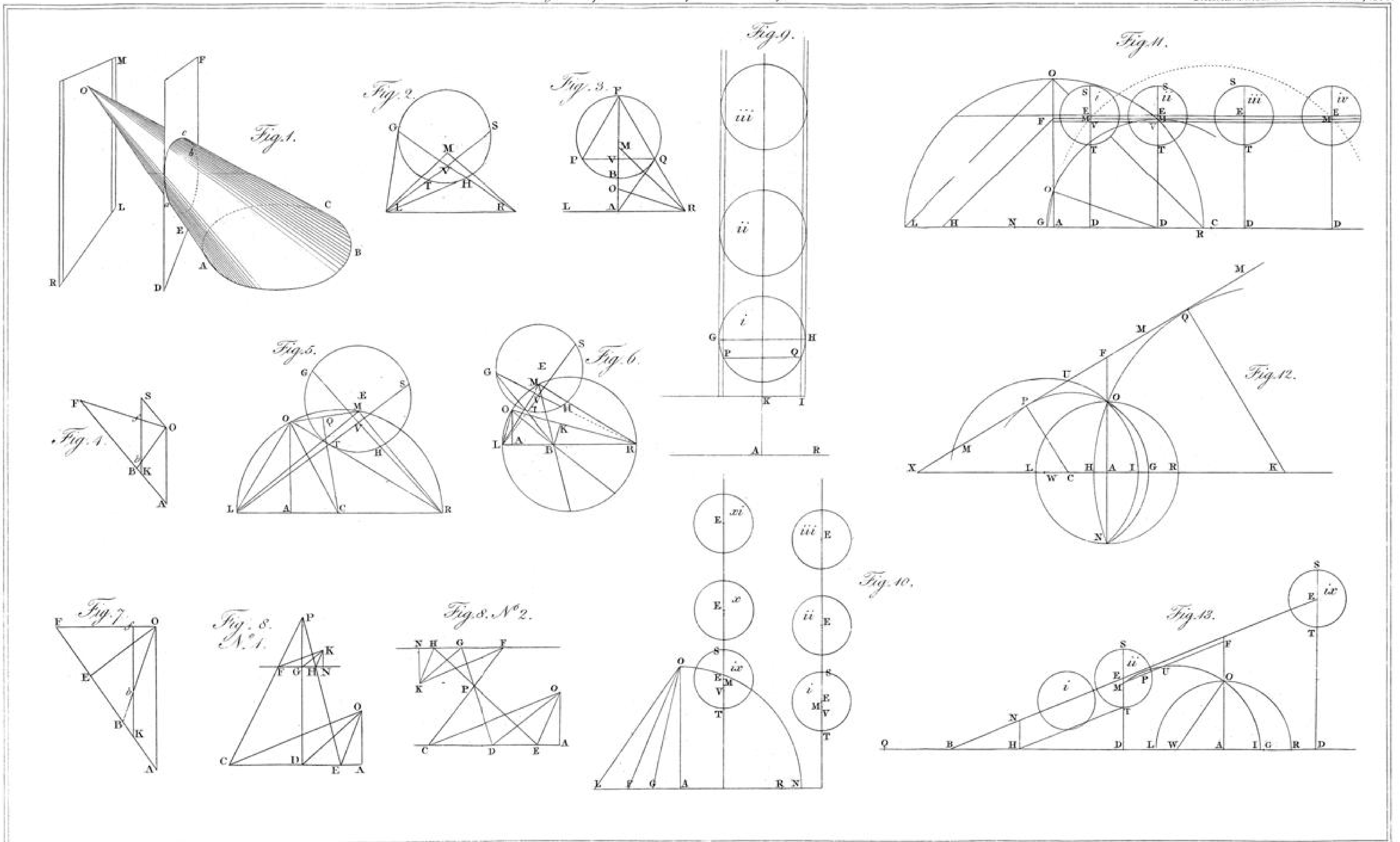
Figures for the Perspective Propositions.



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